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Global Dynamics of 1-D Extended Cellular Automata

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Abstract

Following our algebraic method for investigating information transmission in CA, the global dynamics of the extended $CA[X]$ is studied in relation to that of CA. Computer simulations of 1-D finite cyclic CAs are also presented.

1 Preliminaries

The 1-D CA is defined as usual with the space Z (the set of integers), the neighborhood index N , the state set Q and the local function f and denoted as $CA=(Z,N,Q,f)$. Throughout this paper we assume the 1-D CA with $N=(-1,0,+1)$ and denote simply as $CA=(Q,f)$.

State Set

Q is assumed to be a finite field. Thus $Q=GF(q)$, where $q=p^n$ with prime p and positive integer n . Denote the cardinality of Q as $|Q|$. So $|Q|=q=p^n$.

Local Function

The local function $f : Q \times Q \times Q \rightarrow Q$ can be expressed as follows:

$$f(x, y, z) = u_1 x^{q-1} y^{q-1} z^{q-1} + u_2 x^{q-1} y^{q-1} z^{q-2} + u_3 x^{q-1} y^{q-2} z^{q-1} + \dots \\ + u_{q^3-1} z + u_{q^3}, \text{ where } u_i \in Q \ (1 \leq i \leq q^3). \quad (1)$$

x, y and z assume the state values of the neighboring cells -1 (left), 0 (center) and $+1$ (right), respectively.

Global Map

The configuration set $C = Q^Z$ and the global map $F : C \rightarrow C$ are defined as usual. When a CA is 1-D finite CA of length $n \geq 1$, its configuration is a

word $w \in Q^n$. We confuse the terminologies *word* and *configuration* for the finite and the infinite CAs.

2 Extension of CA

2.1 Information Expressed by X

Let X be a symbol different from those used in equation (1). It stands for an unknown state or the *information* of the cell in CA. We explain first the role of X in the information transmission of the local function using an example.

Example 1.

The binary set $Q = \{0, 1\} = \text{GF}(2)$ and the function $f(x, y, z) = yz + x$.

From the fact that $f(0, 0, 0) = 0$ and $f(1, 0, 0) = 1$, we may write as $f(X, 0, 0) = X$ (i). Similarly we write $f(X, 1, 1) = X + 1 \pmod{2}$ (ii), which comes from the fact that $f(0, 1, 1) = 1$ and $f(1, 1, 1) = 0$. Also we have $f(X, 1, 0) = 1$ from $f(0, 1, 0) = 1$ and $f(1, 1, 0) = 1$ (iii). From the information related point of view, we claim: in cases (i) and (ii) the information X is transmitted to the right, but in case (iii), it vanishes. Note that from the function $X + 1$ (a permutation of Q) we can restore the value of X without any loss of information.

In generalizing the above argument, we consider another polynomial form, which will be called the *information function*.

$$g(X) = a_1 X^{q-1} + a_2 X^{q-2} + \cdots + a_q, \text{ where } a_i \in Q \ (1 \leq i \leq q). \quad (2)$$

g defines a function $Q \rightarrow Q$ and the set of such functions is denoted by $Q[X]$. Evidently $|Q[X]| = q^q$. Note that $Q[X] \supset Q$. The element of $Q[X] \setminus Q$ is called *informative*, while that of Q *constant*.

The polynomial $g(X) \in Q[X]$ is uniquely expressed in the form of *coefficient vector* (a_1, a_2, \dots, a_q) , which is particularly useful for the computer simulation.

When g is a permutation function of Q , its function value, say a , has a unique preimage $g^{-1}(a)$ in the domain Q . Thus a permutation function completely conserves the information of the domain. When g is a constant, however, we can not obtain any information about preimages from the function value. There are intermediate stages of information amount contained

by the information function g . The greater the cardinality of the value set $g(Q)$ is, the greater the information amount is.

2.2 Ring $Q[X]$

The set of information functions $Q[X]$ is characterized as follows. Let $P[X]$ be the polynomial ring over a finite field Q with an indeterminate X . $Q[X]$ be its factor ring by $X^q - X$, i.e. $Q[X] = P[X]/(X^q - X)$. Note that $X^q - X = X(X^{q-1} - 1)$ is a reducible polynomial in $P[X]$. Therefore $Q[X]$ is not a field but a commutative ring with identity [Lidle, et.al.97].

2.3 Extended CA

We define an extended CA $CA[X] = (Q[X], f)$, where $Q[X]$ is the set of cell states. The local function f is expressed by the same polynomial form f as in (Q, f) . The variables x, y and z , however, move in $Q[X]$ instead of Q . That is, $f : Q[X]^3 \rightarrow Q[X]$. The global map is $F : Q[X]^Z \rightarrow Q[X]^Z$. A configuration is called *informative* if a cell state of the configuration is informative. Otherwise it is *constant*. When a $CA[X]$ starts with a constant configuration, its trajectory always behaves in Q^Z .

3 Global Dynamics of $CA[X]$

3.1 Generalities

We investigate the dynamics of a $CA[X]$ in relation to that of CA. Such notions as *injectivity*, *surjectivity*, *reversibility*, *limit sets* and so on are defined and analysed in $CA[X]$ as well.

Substitution

Let a configuration of $CA[X]$ be $w \in Q[X]^Z$. For any $a \in Q$, the word obtained from w by substituting a for the variable X of each cell state $g(X)$ is denoted by w_a . If w is a constant configuration, then by definition $w_a = w$. Substitution is expressed by the (many to one) mapping $\psi_a : w \mapsto w_a$ or $\psi_a(w) = w_a$ for any $a \in Q$.

Example 2. $q = 3$. $GF(3) = \{0, 1, 2\}$. $n = 5$.

If $w = X, 1, X^2 + 1, 0, 0$, then $w_0 = 0, 1, 1, 0, 0$, $w_1 = 1, 1, 2, 0, 0$ and $w_2 = 2, 1, 2, 0, 0$.

Proposition 1.

- (1) $CA[X]$ is injective, if and only if CA is injective.
- (2) $CA[X]$ is surjective, if and only if CA is surjective.

Proof.

- (1) If CA is injective, then for any $a \in Q$ and any pair of distinct configurations w and v , we have $F(w_a) \neq F(v_a)$. Therefore we have $F(w) \neq F(v)$, i.e. $CA[X]$ is injective. The only if part is obvious.
- (2) Let c_X be an arbitrary configuration of $Q[X]^Z$. Since CA is surjective, for any $a \in Q$, there is a constant configuration w such that $F(w) = c_a$. Therefore there is an informative configuration $c'_X \in Q[X]^Z$ such that $\psi_a(c'_X) = w$ and $F(c'_X) = c_X$. So we have the if part of (2). \square

In addition to the above *mathematical* properties pertaining to the global map F , we consider the *informational* properties of CA dynamics. Among others we are interested in the information transmission ability of CA s. When a $CA[X]$ starts with an initial configuration vXw where v and w are constants, the information of X is generally transmitted to the right and left or to the *space direction*. If the trajectory of a $CA[X]$ contains informative configurations forever, then the information is called to be transmitted to the *time direction* without end.

The following proposition is a consequence of Kari's results[Kari94].

Proposition 2.

It is not decidable, whether or not a $CA[X]$ enters a limit set consisting of constant configurations after starting with an initial configuration wXv , where w and v are constant.

The proposition means that as for an arbitrary 1-D CA the ultimate ability of information transmission to the time direction is undecidable[see Nishio99].

A local function f is called *permutive in x* , if $f(Q, y, z) = Q$ for any values of y and z . Similarly permutativity in y (and z) is defined[Hedlund70][Fagnani, et.al.98]. Then we have the following simple result.

Proposition 3.

If a CA is permutive in x or z , then the information is transmitted without end to the space direction.

Proposition 4.

It is undecidable, whether or not a $CA[X]$ transmits the information X to the space direction without end.

Proof.

It was proved undecidable for finite fixed boundary CAs[see Nishio99]. Modification of the proof to fit with infinite CAs is easy.

3.2 Finite CA

Consider a finite CA= (Q, f, n, B) where $n \geq 1$ is the number of cells and B is the boundary condition, *cyclic*, *fixed* and others. Note that the following discussion is not sensitive to the boundary condition.

Cycle and Transient

When a CA starts with a configuration w , its trajectory consists of the finite transient $t(w)$ and the cycle $p(w)$, which follows the transient and repeats itself forever. The lengths of the cycle and the transient are denoted by $\phi(w)$ and $\tau(w)$, respectively.

Computer simulations are shown in **Appendix** for a three state cyclic CA[X]. The system starts with an informative configuration $w = X11111$ in (A) and enters the cycle of length 12 after the transient of length 2. In (B),(C) and (D) it starts with constants $\psi_0(w), \psi_1(w)$ and $\psi_2(w)$, respectively. Note that $\phi(w) = 12 = LCM\{4, 1, 3\}$.

Proposition 5.

- (1) $\phi(w) = LCM\{\phi(w_a) \mid a \in Q\}$.
- (2) $\tau(w) = MAX\{\tau(w_a) \mid a \in Q\}$.

Proof.

The information function $g(X)$ can be represented by a q -tuple of constant vectors $(0, 0, 0, \dots, 0, b_i), b_i \in Q, 1 \leq i \leq q$. In fact $b_i = g(a_i)$ and conversely from a set of q values $b_i, 1 \leq i \leq q$, one can uniquely compute the set of coefficients a_i s which gives $g(X)$. Consequently the dynamics of CA[X] is faithfully simulated by computing separately each dynamics of CA and considering their q -tuples.

- (1) If the trajectory of CA starting with w_a has the cycle length $\phi(w_a)$, then the trajectory of q -tuples of the coefficient vectors has the cycle length of a multiple of each $\phi(w_a)$. It is in fact equal to $LCM\{\phi(w_a) \mid a \in Q\}$.
- (2) When every trajectory of CAs enter the cycle, the q -tuples also become cyclic. Therefore we have (2) of the proposition. \square

We state the following proposition without proof.

Proposition 6.

$\phi(w) = \phi(w_a)$ for any $a \in Q$, if and only if CA[X] enters a cycle consisting

of constant configurations.

Concluding Remarks

The idea has been presented for the basic 1-D CA, though it works for general CAs. The decision problems we treated above asks if or not *any* information is transmitted. The problem asking *how much* information is transmitted is left for further reseach. Thanks are due to Takashi Saito for writing the simulation program of 1-D finite CA[X]s with the language DrScheme.

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(Erratum: Proposition 2 has been proved false by a counter example.)

Appendix: Simulation of CA[X]

$Q = \text{GF}(3)$, cyclic boundary, $n = 6$, $f = xz + y$.

(A) $w = X11111$

time: cell 1 to 6.

```

0 : ((0 1 0) (0 0 1) (0 0 1) (0 0 1) (0 0 1) (0 0 1))
1 : ((0 1 1) (0 1 1) (0 0 2) (0 0 2) (0 0 2) (0 1 1))
2 : ((1 0 2) (0 0 0) (0 2 1) (0 0 0) (0 2 1) (0 0 0))
3 : ((1 0 2) (1 0 2) (0 2 1) (1 1 1) (0 2 1) (1 0 2))
4 : ((0 0 0) (2 0 1) (1 2 0) (2 2 2) (1 2 0) (2 0 1))
5 : ((2 0 1) (2 0 1) (2 2 2) (1 0 2) (2 2 2) (2 0 1))
6 : ((1 0 2) (0 0 0) (0 2 1) (2 1 0) (0 2 1) (0 0 0))
7 : ((1 0 2) (1 0 2) (0 2 1) (0 2 1) (0 2 1) (1 0 2))
8 : ((0 0 0) (2 0 1) (1 2 0) (1 0 2) (1 2 0) (2 0 1))
9 : ((2 0 1) (2 0 1) (2 2 2) (0 1 2) (2 2 2) (2 0 1))
10 : ((1 0 2) (0 0 0) (0 2 1) (1 2 0) (0 2 1) (0 0 0))
11 : ((1 0 2) (1 0 2) (0 2 1) (2 0 1) (0 2 1) (1 0 2))
12 : ((0 0 0) (2 0 1) (1 2 0) (0 1 2) (1 2 0) (2 0 1))
13 : ((2 0 1) (2 0 1) (2 2 2) (2 2 2) (2 2 2) (2 0 1))
14 : ((1 0 2) (0 0 0) (0 2 1) (0 0 0) (0 2 1) (0 0 0))

```

$\tau = 2, \phi = 12$

(B) $w_0 = 011111$

```

0 : ((0 0 0) (0 0 1) (0 0 1) (0 0 1) (0 0 1) (0 0 1))
1 : ((0 0 1) (0 0 1) (0 0 2) (0 0 2) (0 0 2) (0 0 1))
2 : ((0 0 2) (0 0 0) (0 0 1) (0 0 0) (0 0 1) (0 0 0))
3 : ((0 0 2) (0 0 2) (0 0 1) (0 0 1) (0 0 1) (0 0 2))
4 : ((0 0 0) (0 0 1) (0 0 0) (0 0 2) (0 0 0) (0 0 1))
5 : ((0 0 1) (0 0 1) (0 0 2) (0 0 2) (0 0 2) (0 0 1))

```

$\tau = 1, \phi = 4$

(C) $w_1 = 111111$

```

0 : ((0 0 1) (0 0 1) (0 0 1) (0 0 1) (0 0 1) (0 0 1))
1 : ((0 0 2) (0 0 2) (0 0 2) (0 0 2) (0 0 2) (0 0 2))
2 : ((0 0 0) (0 0 0) (0 0 0) (0 0 0) (0 0 0) (0 0 0))
3 : ((0 0 0) (0 0 0) (0 0 0) (0 0 0) (0 0 0) (0 0 0))

```

$\tau = 2, \phi = 1$

(D) $w_2 = 211111$

```

0 : ((0 0 2) (0 0 1) (0 0 1) (0 0 1) (0 0 1) (0 0 1))
1 : ((0 0 0) (0 0 0) (0 0 2) (0 0 2) (0 0 2) (0 0 0))
2 : ((0 0 0) (0 0 0) (0 0 2) (0 0 0) (0 0 2) (0 0 0))
3 : ((0 0 0) (0 0 0) (0 0 2) (0 0 1) (0 0 2) (0 0 0))
4 : ((0 0 0) (0 0 0) (0 0 2) (0 0 2) (0 0 2) (0 0 0))

```

$\tau = 1, \phi = 3$

For example coefficient vector (2,0,1) means $2X^2 + 1$.